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ON THE GERSTENHABER–RAUCH PRINCIPLE

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ABSTRACT

Suppose $f^*(z)$ is a K^*-qc self-homeomorphism of the unit disk U, where K^* is the minimum possible value among all qc mappings of U with the same boundary values as f^*. It is known that K^* can be calculated by a variational principle involving mappings of U harmonic with respect to admissible weight functions. We examine the weight functions that correspond to the case when the extremum for the variational principle is attained, and characterize the corresponding mappings f^*.

0. Introduction

For a quasiconformal (homeomorphic) or qc mapping $w = f(z)$ of the unit disk $U = \{|z| < 1\}$ onto $U = \{|w| < 1\}$ we use the standard notations,

$$\kappa_f(z) = \frac{f_{\bar z}}{f_z}, \quad k_f(z) = |\kappa_f(z)|, \quad D_f(z) = \frac{1 + k_f(z)}{1 - k_f(z)} = \frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|},$$

$$k[f] = \operatorname*{ess\,sup}_{z \in U} k_f(z), \quad K[f] = \operatorname*{ess\,sup}_{z \in U} D_f(z) = \frac{1 + k[f]}{1 - k[f]}.$$

If H is a homeomorphism of ∂U onto ∂U we denote by $Q(H)$ the class of qc mappings of U onto itself with boundary values H. In order to avoid triviality we assume that $Q(H)$ is non-empty, and that H is not the boundary restriction of a conformal mapping. The homeomorphism H then determines the *extremal* maximal dilatation $K^* > 1$, defined as

$$K^* = \inf_{f \in Q(H)} K[f]. \tag{0.1}$$

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If $f \in Q(H)$ and if $\rho(w) \geqq 0$ ($|w|<1$) is a measurable function of $w = u + iv$ with $\iint_{|w|<1} \rho(w)du\,dv = 1$, one defines the Douglas–Dirichlet functional as

$$(0.2)\qquad \mathcal{D}_\rho[f] = \iint_U \rho(f(z))(|f_z|^2 + |f_{\bar z}|^2)dx\,dy.$$

If we let $g(w) = f^{-1}(w)$, it is easy to see that $\mathcal{D}_\rho$ can also be expressed as

$$(0.3)\qquad \begin{aligned} \mathcal{D}_\rho[f] &= \iint_{|w|<1} \rho(w)\frac{|g_w|^2+|g_{\bar w}|^2}{|g_w|^2-|g_{\bar w}|^2}\,du\,dv \\ &= \frac{1}{2}\iint_{|w|<1} \rho(w)\left[D_g(w)+\frac{1}{D_g(w)}\right]du\,dv. \end{aligned}$$

In particular, it is clear from (0.3) that $\mathcal{D}_\rho[f] < \infty$ for all $f \in Q(H)$ and for any admissible $\rho(w)$.

It was proved in [3] that

$$(0.4)\qquad \sup_\rho \inf_{f \in Q(H)} \mathcal{D}_\rho[f] = \frac{1}{2}\left(K^* + \frac{1}{K^*}\right).$$

We refer to (0.4) as the Gerstenhaber–Rauch principle, because an analogous conjecture (for compact surfaces rather than for U) was formulated by them [1]. The potential attractiveness of the extremal problem (0.4), as realized by Gerstenhaber and Rauch (who, however, restricted themselves mainly to formal aspects), is that the inside extremal problem,

$$\inf_f \mathcal{D}_\rho[f],$$

leads to the simple Euler condition (see Lemma 1.1, below, for details)

$$(0.5)\qquad \rho(f(z))f_z\overline{f_{\bar z}} = \varphi(z) = \text{holomorphic in } U.$$

One refers to condition (0.5) as the condition that $f(z)$ is *harmonic* in U *with respect to the weight function* $\rho(w)$. There is a large literature on harmonic mappings. (See the monograph [2] and its bibliography.)

We shall be concentrating here on the question of when in the operation sequence $\sup_\rho \inf_f$ of (0.4) the inf and sup are simultaneously attained. We refer to ρ_0 as *maximal* for the Douglas–Dirichlet functional for H if $\rho_0(w)$ ($|w|<1$) is an admissible weight function, and if

$$(0.6)\qquad \inf_f \mathcal{D}_{\rho_0}[f] = \frac{1}{2}\left(K^* + \frac{1}{K^*}\right).$$

If $f_0 \in Q(H)$, we call f_0 *minimal* for $\mathscr{D}_\rho$ if $\mathscr{D}_\rho[f_0] = \inf_{f \in Q(H)} \mathscr{D}_\rho[f]$. On the other hand, the term *extremal mapping* for H shall be reserved for an element $f^* \in Q(H)$ for which $K[f^*] = K^*$. It is well known that $Q(H)$ must contain at least one extremal mapping, but it is also known that there may be more than one.

The simplest and most important general example of extremal mappings of U onto U are the so-called *Teichmüller mappings* corresponding to holomorphic functions of finite norm. These are mappings $f_0(z)$ such that for some *constant* k_0, $0 < k_0 < 1$, one has

$$\kappa_{f_0}(z) = k_0 \frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|} \qquad (|z| < 1), \tag{0.7}$$

where $\varphi_0(z)$ belongs to the class $\mathscr{B}(U)$ of functions *holomorphic* in U with finite non-vanishing L^1 norm

$$0 < \|\varphi_0\| = \iint_U |\varphi_0(z)|\, dx\, dy < \infty.$$

For every mapping of this type there exists [6] an associated holomorphic function $\psi_0(w)$ determined by k_0 and φ_0, with

$$0 < \|\psi_0(w)\| = \iint |\psi_0(w)|\, du\, dv < \infty,$$

such that, for $g_0 = f_0^{-1}$,

$$\kappa_{g_0}(w) = -k_0 \frac{\overline{\psi_0(w)}}{|\psi_0(w)|} \qquad (|w| < 1). \tag{0.8}$$

It is known [6] that if f_0 is of type (0.7), and if H_0 denotes the induced boundary homeomorphism,

$$H_0 = f_0|_{\partial U},$$

then f_0 is an extremal mapping for H_0, and moreover it is the only extremal mapping for H_0.

Whether or not a maximal weight function exists depends on H. It turns out (Theorem 2.1) that the only possible maximal weight function ρ_0 is $\rho_0(w) = |\psi_0(w)|$, where $\psi_0(w) \in \mathscr{B}(U)$, and that any H which allows such a maximal weight function must be the boundary correspondence induced by a corresponding Teichmüller mapping f_0. Moreover (Theorem 4.1), for such an H, f_0 is the only minimal mapping. Thus, we have a complete description of when the operation "sup inf" in the Gerstenhaber–Rauch principle can be replaced by "max min". Special cases of these results were previously obtained in [4].

Section 3 is devoted to an inequality (Theorem 3.1) of independent interest, including a complete study of when equality can occur. The latter part of the result is needed in the proof of Theorem 4.1.

1. Harmonic mappings

Let $f(z)$ be quasiconformal with domain U, and let $F(z)$ be a qc map of U onto U. A straightforward computation leads to the transformation formula

$$\begin{aligned}\mathcal{D}_\rho[f\circ F^{-1}] = \mathcal{D}_\rho[f] + 2\iint_U \frac{|\kappa_F(z)|^2}{1-|\kappa_F(z)|^2}\rho(f(z))(|f_z|^2+|f_{\bar z}|^2)dx\,dy \\ -4\,\mathrm{Re}\iint_U \frac{\kappa_F(z)}{1-|\kappa_F(z)|^2}\varphi(z)dx\,dy,\end{aligned} \tag{1.1}$$

where

$$\varphi(z) = \rho(f(z))f_z\overline{f_{\bar z}}, \qquad z\in U. \tag{1.2}$$

(In particular, it is clear that if F is a Mobius transformation, then $\mathcal{D}_\rho[f\circ F^{-1}] = \mathcal{D}_\rho[f]$.) We have

$$\|\varphi\| = \iint_U |\varphi(z)|\,dx\,dy = \iint_{|w|<1}\rho(w)\frac{|f_z||f_{\bar z}|}{|f_z|^2-|f_{\bar z}|^2}du\,dv \leqq \frac{k[f]}{1-k[f]^2}.$$

Thus, $\varphi\in L^1(U)$.

The following variational lemma occurs in formal form already in [1].

LEMMA 1.1. *Let $\rho(w)$, H be given. Suppose there exists $f_0\in Q(H)$ such that f_0 is minimal for $\mathcal{D}_\rho$:*

$$\mathcal{D}_\rho[f_0] = \inf_{f\in Q(H)}\mathcal{D}_\rho[f]. \tag{1.3}$$

Then

$$\varphi_0(z) = \rho(f_0(z))f_{0z}\overline{f_{0\bar z}}\in\mathcal{B}(U); \tag{1.4}$$

in particular, $f_0(z)$ is harmonic in U with respect to ρ.

PROOF. Let $\lambda(z)$ be a complex-valued function of class C^1 in U with compact support, and let

$$F_\varepsilon(z) = z+\varepsilon\lambda(z).$$

Say, $|\lambda_z|+|\lambda_{\bar z}|\leqq M$. Then, if $0\leqq|\varepsilon|<1/M$, $F_\varepsilon(z)$ is evidently a qc mapping of U onto U whose boundary values are the identity mapping. Moreover,

$$\kappa_{F_\varepsilon}(z)=\frac{\varepsilon\lambda_{\bar z}}{1+\varepsilon\lambda_z}\ .$$

Applying (1.1) to $f_0\circ F^{-1}$ we see that a necessary condition for (1.3) is that

$$\iint_U \lambda_{\bar z}\varphi_0(z)\,dx\,dy=0.$$

Since we already know that $\varphi_0\in L^1(U)$, the conclusion that $\varphi_0\in\mathscr{B}(U)$ follows by Weyl's Lemma.

For the qc mapping f satisfying (1.4) we see, by taking the absolute value of both sides, that

$$\kappa_{f_0}(z)=k_f(z)\frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|}\,,\qquad \text{where } \varphi\in\mathscr{B}(U). \tag{1.5}$$

Such a mapping is referred to as a *generalized* Teichmüller mapping corresponding to a holomorphic function of finite norm. Evidently, every qc harmonic mapping is of this sort.

2. Maximal weight functions

If H is a boundary correspondence induced by a Teichmüller mapping (0.7), and if $\rho_0(w)=|\psi_0(w)|$, where $\psi_0(w)$ is the associated holomorphic function, we have $K^*=(1+k_0)/(1-k_0)$, and one may verify that (0.6) is satisfied; that is, ρ_0 is maximal. We will show that the following converse holds.

THEOREM 2.1. *Let H be given. Suppose there exists a weight function $\rho_0(w)$, such that* (0.6) *holds. Then*

(i) *There is a unique extremal mapping f^* in $Q(H)$. It is a Teichmüller mapping corresponding to a holomorphic function $\varphi^*(z)$ of class $\mathscr{B}(U)$:*

$$\kappa_{f^*}(z)=k^*\frac{\overline{\varphi^*(z)}}{|\varphi^*(z)|}\,,\qquad k^*=\frac{K^*-1}{K^*+1}\,, \tag{2.1}$$

and satisfies

$$\mathscr{D}_{\rho_0}[f^*]=\inf_f\,\mathscr{D}_{\rho_0}[f]. \tag{2.2}$$

(ii) *The weight function* $\rho_0(w)$ *is uniquely determined; namely,*

$$\rho_0(w) = |\psi^*(w)|,$$

where $\psi^*(w)$ ($\|\psi^*\| = 1$) *is the holomorphic function of* w *determined by* φ^* *and* k^* *with the property that*

$$\kappa_{f^{*-1}}(w) = -k^* \frac{\overline{\psi^*(w)}}{|\psi^*(w)|}. \tag{2.3}$$

PROOF. (i) Let f^* be an extremal for $Q(H)$, and let $g^* = f^{*-1}$. For every weight function $\rho(w)$ we have

$$\mathcal{D}_\rho[f^*] = \frac{1}{2}\iint_{|w|<1} \rho(w)\left[D_{g^*}(w) + \frac{1}{D_{g^*}(w)}\right] du\, dv \leqq \frac{1}{2}\left(K^* + \frac{1}{K^*}\right).$$

Therefore, in view of (0.6),

$$\frac{1}{2}\left(K^* + \frac{1}{K^*}\right) \geqq \mathcal{D}_{\rho_0}[f^*] \geqq \inf_f \mathcal{D}_{\rho_0}[f] = \frac{1}{2}\left(K^* + \frac{1}{K^*}\right).$$

So, equality must hold throughout:

$$\mathcal{D}_{\rho_0}[f^*] = \inf_{f \in Q(H)} \mathcal{D}_{\rho_0}[f] = \frac{1}{2}\left(K^* + \frac{1}{K^*}\right), \tag{2.4}$$

and, in particular,

$$D_{g^*}(w) = K^* \quad \text{at a.a. points where } \rho_0(w) \neq 0. \tag{2.5}$$

Let

$$\varphi^*(z) = \rho_0(f^*(z)) f_z^* \overline{f_{\bar z}^*}. \tag{2.6}$$

By (2.4) and Lemma 1.1, $\varphi^*(z) \in \mathcal{B}(U)$, with

$$\|\varphi^*\| = \iint_{|w|<1} \rho_0(w) \frac{|f_z^*|\,|f_{\bar z}^*|}{|f_z^*|^2 - |f_{\bar z}^*|^2}\, du\, dv = \iint_{|w|<1} \rho_0(w) \frac{|\kappa_{g^*}(w)|}{1 - |\kappa_{g^*}(w)|^2}\, du\, dv$$

$$= \frac{k^*}{1 - k^{*2}}.$$

Since $\varphi^*(z)$ is holomorphic and $\|\varphi^*\| \neq 0$, we see from (2.6) that

$$\rho_0(w) \neq 0 \quad \text{a.e.}$$

So, in view of (2.5),

$$D_{g^*}(w) = K^* \qquad \text{a.e. for } |w| < 1. \tag{2.7}$$

Now, by (2.6),

$$\kappa_{f^*}(z) = k_{f^*}(z)\frac{\overline{\varphi^*(z)}}{|\varphi^*(z)|}.$$

Since, by (2.7), $k_{f^*}(z) = k^*$ a.e., the proof of (i) is complete.

(ii) Locally,

$$w = f^*(z) = \Psi^{*-1}[a\Phi^* + ak^*\overline{\Phi^*}],$$

where $\varphi^*(z) = \Phi^{*\prime}(z)^2$, $\psi^*(w) = \Psi^{*\prime}(w)^2$; the constant $a > 0$ can be chosen to normalize $\|\varphi^*\|$ and $\|\psi^*\|$ as desired. Therefore,

$$J(w/z) = |f^*_z|^2 - |f^*_{\bar z}|^2 = a^2(1-k^{*2})|\varphi^*(z)|/|\psi^*(w)|,$$

and

$$\|\psi^*\| = \iint_{|z|<1} |\psi^*(w)| J(w/z)\,dx\,dy = a^2(1-k^{*2})\|\varphi^*\| = a^2k^*.$$

By (2.6), therefore,

$$\varphi^*(z) = \rho_0(w)\frac{a\Phi^{*\prime}(z)}{\Psi^{*\prime}(w)}\frac{ak^*\Phi^{*\prime}(z)}{\Psi^{*\prime}(w)} = a^2k^*\rho_0(w)\frac{\varphi^*(z)}{|\psi^*(w)|}.$$

Thus,

$$|\psi^*(w)| = a^2k^*\rho_0(w) = \rho_0(w),$$

providing we choose the normalizing parameter a as

$$a = 1/\sqrt{k^*}.$$

We note that a weight function which vanishes on a set of positive measure cannot possibly be maximal.

3. A mean-dilatation inequality

Suppose f and f_0 are qc mappings of U onto U which agree on ∂U, and suppose $\kappa_{f_0}(z)$ has the form (0.7). In Section 0, we referred to the known fact that in this case

$$\operatorname*{ess\,sup}_{|z|<1} D_f(z) \geqq K_0,$$

and that equality implies that $f = f_0$. The following result is considerably stronger.

THEOREM 3.1. *Let $w = f_0(z)$ be a Teichmüller mapping of $\{|z|<1\}$ onto $\{|w|<1\}$ with complex dilatation of the form* (0.7). *Let $f(z)$ be a qc mapping of $\{|z|<1\}$ whose boundary values agree with those of $f_0(z)$. Then*

$$\iint_{|z|<1} |\varphi_0(z)| D_f(z) dx\, dy \geqq K_0 \iint_{|z|<1} |\varphi_0(z)| dx\, dy \qquad \left(K_0 = \frac{1+k_0}{1-k_0}\right). \tag{3.1}$$

Equality holds in (3.1) *if and only if $f = f_0$.*

PROOF. Consider the inverse mappings $g_0(w) = f_0^{-1}(w)$, $g(w) = f^{-1}(w)$, $|w|<1$. $\kappa_{g_0}(w)$ is given by (0.8). Except for the purely notational change of replacing z by w and $\varphi_0(z)$ by $-\psi_0(w)$, (3.1) is equivalent to establishing that

$$\iint_{|w|<1} |\psi_0(w)| D_g(w) du\, dv \geqq K_0 \iint_{|w|<1} |\psi_0(w)| du\, dv. \tag{3.1'}$$

We will prove that (3.1′) is valid, and that equality in (3.1′) can occur only when $g = g_0$.

To derive (3.1′) we use a basic inequality derived in [5]. Let $\varphi(z)$ be any element of $\mathcal{B}(U)$, $\|\varphi\| = 1$, and let κ_{f_0} and κ_g denote $\kappa_{f_0}(z)$ and $\kappa_g(f_0(z))$, respectively. Also, let $p = f_{0z}$, $q = f_{0\bar{z}}$. According to relation (1.2.8) of [5], adapted to our current situation,

$$1 \leqq \iint_{|z|<1} |\varphi| \frac{|1 + (\bar{p}/p)\kappa_g \overline{\kappa_{f_0}} - [\kappa_{f_0} + (\bar{p}/p)\kappa_g](\varphi/|\varphi|)|^2}{(1-|\kappa_g|^2)(1-|\kappa_{f_0}|^2)} dx\, dy. \tag{3.2}$$

An elementary computation shows that for any qc mapping g and its inverse f

$$\frac{g_{\bar{w}}}{g_w} = -\frac{f_{\bar{z}}}{\bar{f}_z} \frac{\bar{f}_{\bar{z}}}{f_z} \qquad (w = f(z)).$$

Thus, in (3.2), we can replace $\bar{p}/p$ by

$$\frac{\bar{p}}{p} = -\frac{\kappa_{f_0}(z)}{\kappa_{g_0}(f_0(z))}. \tag{3.3}$$

In the present case we apply (3.2), (3.3) with $\kappa_{f_0}(z)$ and $\kappa_{g_0}(w)$ given by (0.7) and (0.8). As the function $\varphi(z)$ of (3.2), we use $\varphi(z) = \varphi_0(z)$. We can actually postulate that both normalizations

$$\|\varphi_0\| = \iint_{|z|<1} |\varphi_0(z)| dx\, dy = 1 \quad \text{and} \quad \|\psi_0\| = \iint_{|w|<1} |\psi_0(w)| du\, dv = 1 \tag{3.4}$$

hold if we represent $w = f_0(z)$ locally as

$$w = f_0(z) = \Psi_0^{-1}\left[\sqrt{K_0}\operatorname{Re}\Phi_0(z) + \frac{i}{\sqrt{K_0}}\operatorname{Im}\Phi_0(z)\right], \quad \varphi_0 = \Phi_0'^2, \quad \psi_0 = \Psi_0'^2. \tag{3.5}$$

After simplification, (3.2) assumes the form

$$K_0 \leqq \iint_{|z|<1} |\varphi_0(z)| \frac{\left|1 + k_0 \frac{\kappa_g(w)}{\kappa_{g_0}(w)}\right|^2}{1 - |\kappa_g(w)|^2} \, dx\, dy \qquad (w = f_0(z)), \tag{3.6}$$

or, since (3.5) implies

$$|\psi_0(w)|\, du\, dv = |\varphi_0(z)|\, dx\, dy \qquad (w = f_0(z)),$$

the equivalent form

$$K_0 \leqq \iint_{|w|<1} |\psi_0(w)| \frac{\left|1 + k_0 \frac{\kappa_g(w)}{\kappa_{g_0}(w)}\right|^2}{1 - |\kappa_g(w)|^2} \, du\, dv. \tag{3.7}$$

From (3.7) we immediately deduce assertion (3.1′).

We shall now consider the consequences of equality in (3.1′). From the fact that equality must then occur in (3.7) it follows that

$$\kappa_g(w) = -k_g(w) \frac{\overline{\psi_0(w)}}{|\psi_0(w)|}. \tag{3.8}$$

Our objective is to prove that

$$k_g(w) = k_0. \tag{3.9}$$

One cannot conclude that (3.9) holds merely from (3.8) and the fact that g and g_0 agree when $|w| = 1$. Instead, we go back to [5] to examine the consequences of equality in (3.2). The derivation of (3.2) in [5] is based on a length–area method involving lengths of trajectories of quadratic differentials in Teichmüller metrics. A straightforward analysis of the derivation indicates that equality in (3.2) implies the following: *Let β be an orthogonal trajectory of the quadratic differential*[†] *$\psi_0(w)dw^2$; that is, let β be a curve such that*

$$\psi_0(w)dw^2 < 0, \qquad w \in \beta.$$

Then the images, $g(\beta)$ and $g_0(\beta)$, under the two mappings $z = g(w)$ and $z = g_0(w)$, must coincide:

$$g(\beta) = g_0(\beta),$$

[†] An exposition of facts about quadratic differentials used in the remainder of this section may be found in [7].

and the common image must be an orthogonal trajectory of the quadratic differential $\varphi_0(z)dz^2$.

Consider the directional derivative

$$\left|\frac{dz}{dw}\right|_\theta$$

of the mapping $z = g(w)$ at the point w, in the direction θ, which exists at a.a. points w, in all directions. Let us assume, first, that $k_g(w) \neq 0$ at the point being considered. In view of (3.8),

$$\max_\theta \left|\frac{dz}{dw}\right|_\theta$$

is uniquely attained (at a.a. w) in the direction tangent to the (unique) β-curve passing through w. On the other hand, if we consider the α-curves or ("horizontal") trajectories of $\psi_0(w)dw^2$, defined by

$$\psi_0(w)dw^2 > 0, \qquad w \in \alpha,$$

it is evident, by (3.8), that these have the following two properties. Firstly, at a.a. points there is a unique α-curve and a unique β-curve which cut each other orthogonally; secondly, $\min_\theta |dz/dw|_\theta$ is uniquely attained in the direction tangent to the α-curve.

For the mapping $w = f(z)$ which is inverse to $z = g(w)$, directions in which $|dw/dz|$ is respectively maximum and minimum correspond to the directions in which $|dz/dw|$ is respectively minimum or maximum. Since we saw above that β-curves are mapped by g onto orthogonal trajectories of $\varphi_0(z)dz^2$, and since at a.a. points z the trajectories of $\varphi_0(z)dz^2$ cut the orthogonal trajectories of $\varphi_0(z)dz^2$ at right angles, it follows that $z = g(w)$ *maps every* α-*curve onto a trajectory of* $\varphi_0(z)dz^2$.

The conclusion that directions tangent to α-curves are mapped by g onto directions dz for which $\varphi_0(z)dz^2 > 0$ also holds at a.a. points w where $k_g(w) = 0$, since g is angle-preserving at a.a. such points. So, we can conclude that *every* α-curve is mapped by g onto a trajectory $\varphi_0(z)dz^2 > 0$. Of course, the Teichmüller mapping $g_0(w)$ also maps α-curves onto trajectories $\varphi_0(z)dz^2 > 0$; thus, *for every* α-*curve, both* $g(\alpha)$ *and* $g_0(\alpha)$ *are trajectories of* $\varphi_0(z)dz^2$. We claim, in fact, that $g(\alpha)$ *and* $g_0(\alpha)$ *are the same.* This follows, since g and g_0 have the same boundary values, with the help of the following two facts:

(a) Almost all trajectories form cross cuts of U of finite length in the Teichmüller metric.

(b) In view of the unique geodetic properties of trajectories these cross cuts are uniquely determined by their pair of endpoints.

In summary, therefore, we have seen that when equality holds in (3.1′), then at a.a. points w the images of the (unique) α-curves and β-curves through w are the same under the two mappings g and g_0, and are, respectively, trajectories and orthogonal trajectories of $\varphi_0(z)dz^2$. Due to the geodetic properties, a trajectory and an orthogonal trajectory of a quadratic differential holomorphic in U can intersect in at most one point. So at each of these points w, $g(w)$ is uniquely determined as $g_0(w)$. Clearly, therefore, $g = g_0$ on its domain U, and the proof is complete.

4. Unique minimal mapping property

THEOREM 4.1. *Let H be given. If there exists a maximal weight function $\rho_0(w)$ (i.e., a weight function for which* (0.6) *holds) then there exists one and only one mapping $f_0 \in Q(H)$, such that $\mathcal{D}_{\rho_0}[f_0] = \inf_{f \in Q(H)} \mathcal{D}_{\rho_0}[f]$. f_0 is a Teichmüller mapping corresponding to a holomorphic function of finite norm, and hence an extremal mapping for H.*

PROOF. By Theorem 2.1, there exists a uniquely determined holomorphic function $\psi^*(w)$ in $\mathcal{B}(U)$, with the property that

$$\rho_0(w) = |\psi^*(w)|, \tag{4.1}$$

and there is a unique extremal map f^* for H. The map f^* is a Teichmüller map, corresponding to a holomorphic function $\varphi^*(z)$ of finite norm, satisfying (2.1) and (2.3).

Let $f_0 = f^*$. According to (2.2), f_0 is *minimal* for $\mathcal{D}_{\rho_0}$. Our objective is to show that f_0 is the *only* minimal mapping for $\mathcal{D}_{\rho_0}$ in $Q(H)$.

Suppose $f \in Q(H)$, and

$$\mathcal{D}_{\rho_0}[f] = \mathcal{D}_{\rho_0}[f_0] = \frac{1}{2}\left(K^* + \frac{1}{K^*}\right).$$

Let $g = f^{-1}$. By (0.3), (4.1),

$$\iint_{|w|<1} |\psi_0(w)| \frac{1 + k_g(w)^2}{1 - k_g(w)^2}\, du\, dv = \frac{1}{2}\left(K^* + \frac{1}{K^*}\right) = \frac{1 + k^{*2}}{1 - k^{*2}}. \tag{4.2}$$

Now

$$\frac{1 + s^2}{1 - s^2} = G\left(\frac{1+s}{1-s}\right), \quad \text{where } G(t) = \frac{1}{2}\left(t + \frac{1}{t}\right).$$

Since $G(t)$ is a convex increasing function of t, $t \geqq 1$, we conclude from (4.2), by Jensen's inequality, that

$$\iint_{|w|<1} |\psi_0(w)| \frac{1+k_g(w)}{1-k_g(w)} \, du \, dv \leqq \frac{1+k^*}{1-k^*} \, .$$

Therefore, by Theorem 3.1, the above must be an equality, and we can conclude by Theorem 3.1 that $g = g_0 = f_0^{-1}$. Thus, $f = f_0$.

References

1. Murray Gerstenhaber and H. E. Rauch, *On extremal quasiconformal mappings*, Proc. Natl. Acad. Sci. U.S.A. **40** (1954), 808–812, 991–994.

2. Jürgen Jost, *Harmonic Maps between Surfaces*, Lecture Notes in Mathematics No. 1062, Springer-Verlag, 1984.

3. Edgar Reich, *On the variational principle of Gerstenhaber and Rauch*, Ann. Acad. Sci. Fenn. **10** (1985), 469–475.

4. Edgar Reich, *Harmonic mappings and quasiconformal mappings*, J. Analyse Math. **46** (1986), 239–245.

5. Edgar Reich and Kurt Strebel, *Extremal quasiconformal mappings with given boundary values*, in *Contributions to Analysis — A Collection of Papers Dedicated to Lipman Bers*, Academic Press, 1974, pp. 375–391.

6. Kurt Strebel, *Zur Frage der Eindeutigkeit extremaler quasikonformer Abbildungen des Einheitskreises II*, Comment. Math. Helv. **39** (1964), 77–89.

7. Kurt Strebel, *Quadratic Differentials*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 5, Springer-Verlag, Berlin, 1984.